

# VECTOR-VALUED EXTENSIONS FOR FRACTIONAL INTEGRALS OF LAGUERRE EXPANSIONS

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ABSTRACT. We prove some vector-valued inequalities for fractional integrals in the setting of Laguerre expansions of Hermite type and convolution type. We use our result for the convolution case to study the fractional integrals related to the harmonic oscillator in mixed norm spaces.

## 1. INTRODUCTION

Consider the fractional integral

$$I_\sigma f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{\|x - y\|^{n-\sigma}} dy, \quad x \in \mathbb{R}^n, \quad 0 < \sigma < n,$$

defined for any function  $f$  for which the above integral is convergent. Then, with an appropriate constant  $c_\sigma$ ,

$$(-\Delta)^{-\sigma/2} f = c_\sigma I_\sigma f, \quad f \in \mathcal{S}(\mathbb{R}^n),$$

where  $\Delta$  is the standard Laplacian in  $\mathbb{R}^n$ , and the negative power is defined in  $L^2(\mathbb{R}^n, dx)$  by means of the Fourier transform, see [19, Ch. 5].

The classical Hardy-Littlewood-Sobolev (see, e.g., [8, 19]) inequality establishes that

$$\|I_\sigma f\|_{L^q(\mathbb{R}^n, dx)} \leq C \|f\|_{L^p(\mathbb{R}^n, dx)},$$

when  $1 < p < \frac{n}{\sigma}$  and  $\frac{1}{q} = \frac{1}{p} - \frac{\sigma}{n}$ . Moreover, in the case  $(p, q) = (1, \frac{n}{n-\sigma})$  a weak type inequality holds. A weighted version of the Hardy-Littlewood-Sobolev was given in [20].

Numerous analogues of the Euclidean fractional integral operator were investigated in various settings for the last decades. For instance, B. Muckenhoupt and E. M. Stein analyzed the topic for ultraspherical expansions in [17]. The one dimensional Hermite and Laguerre function expansions have been considered in [10, 11, 12]. Recently, B. Bongioanni and J. L. Torrea [2] obtained estimates for the negative powers of the multidimensional harmonic oscillator. Fractional integrals for the multidimensional Laguerre expansions (or negative powers of the corresponding second order differential operators) have been treated by A. Nowak and K. Stempak in [14]. They analyzed  $L^p - L^q$  estimates (with and without weights) for the expansions related to Laguerre functions of Hermite type and Laguerre functions of convolution type. A complete and exhaustive study of fractional integrals for Jacobi and Fourier-Bessel expansions has been developed recently in [13]. Moreover, a vector-valued extension in the Jacobi case has been done in [5].

The aim of this paper is the extension of  $L^p - L^q$  mapping properties concerning fractional integrals (or negative powers) related to second order differential operators of Laguerre type. Namely, our target will be the proof of vector-valued extensions of some results given in [14]. We will deal with vector-valued inequalities of the form

$$\left\| \left( \sum_{j=0}^{\infty} |T_j f_j|^r \right)^{1/r} \right\|_{L^p(X, d\mu)} \leq C \left\| \left( \sum_{j=0}^{\infty} |f_j|^r \right)^{1/r} \right\|_{L^q(X, d\mu)},$$

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where  $\{T_j\}_{j \geq 0}$  is a sequence of operators defined on a  $\sigma$ -finite measure space  $(X, d\mu)$ . We will also consider weighted vector-valued inequalities.

If we denote, respectively, by  $(L_\alpha^H)^{-\sigma}$  and  $(L_\alpha)^{-\sigma}$  the fractional operators for Laguerre expansions of Hermite type and for Laguerre expansions of convolution type (see Section 2 for definitions), we are interested in the analysis of vector-valued inequalities for the sequences of operators  $\{(L_{\alpha+aj}^H)^{-\sigma}\}_{j \geq 0}$  and  $\{(L_{\alpha+aj})^{-\sigma}\}_{j \geq 0}$ , where  $a$  is positive real parameter. The meaning of  $\alpha + aj$  will be explained below. For Laguerre expansions of Hermite type we will consider the space  $(\mathbb{R}_+^n, dx)$ , with  $dx$  the Lebesgue measure and our approach will be similar to [14]. In our case, we will prove an estimate for the kernels of the operators  $(L_{\alpha+aj}^H)^{-\sigma}$  independent of the parameter  $j$ . Then a general result of vector-valued extensions will do the work. We will also include some potential weights. On the other hand, we will deal with fractional integrals related to Laguerre expansions of convolution type, where the considered space will be  $(\mathbb{R}_+, d\mu_\alpha)$  with  $d\mu_\alpha = x^{2\alpha+1} dx$ . However, the way to prove the corresponding vector-valued inequality will be close to the ideas in [2]. Observe that in [2], the authors consider Hermite expansions. The argument given in [14] to treat the convolution type setting. The reason is the following: the constants appearing in this case involve Gamma functions whose log-convexity makes these constants increase with no control at all.

As an application of our result about Laguerre expansions of convolution type, we will analyze the fractional integral operator related to spherical eigenfunctions of the harmonic oscillator. The result in [2] deal with the same topic but for the eigenfunctions of the harmonic oscillator in cartesian coordinates. In our situation we consider the eigenfunctions obtained by using spherical coordinates (this is the reason for the name *spherical eigenfunctions*). In the spherical case, the eigenfunctions are products of Laguerre functions and spherical harmonics. The most adequate spaces to deal with this system are the mixed norm spaces  $L^{p,2}(\mathbb{R}^n, r^{n-1} dr d\sigma)$  (see Section 3 for definition). These spaces appear frequently in harmonic analysis when the spherical harmonics are involved. The papers [18, 6, 4, 3] contain good examples of the use of these mixed norm spaces.

The organization of the paper is the following. In Section 2 we introduce some definitions related to Laguerre systems and establish our results about boundedness of the fractional integrals related. Section 3 contains our analysis of the fractional integrals for the harmonic oscillator in mixed norm spaces. In Section 4 and Section 5 we give the proofs of the results given in Section 2. Finally, in Section 6 we show the proofs of some technical results used along the paper.

## 2. DEFINITIONS AND MAIN RESULTS

Let  $\alpha = (\alpha_1, \dots, \alpha_n) \in (-1, \infty)^n$  be a multi-index and  $x, y \in \mathbb{R}_+^n$ ,  $n \geq 1$ . The Laguerre function on  $\mathbb{R}_+^n$  is the tensor product

$$\varphi_k^\alpha(x) = \varphi_{k_1}^{\alpha_1}(x_1) \cdot \dots \cdot \varphi_{k_n}^{\alpha_n}(x_n), \quad x = (x_1, \dots, x_n) \in \mathbb{R}_+^n,$$

where  $\varphi_{k_i}^{\alpha_i}(x_i)$  are the one dimensional Laguerre functions

$$\varphi_{k_i}^{\alpha_i}(x_i) = \left( \frac{2\Gamma(k_i + 1)}{\Gamma(k_i + \alpha_i + 1)} \right)^{1/2} L_{k_i}^{\alpha_i}(x_i^2) x_i^{\alpha_i+1/2} e^{-x_i^2/2}, \quad x_i > 0, \quad i = 1, \dots, n$$

and  $L_{k_i}^{\alpha_i}$  denotes the Laguerre polynomial of degree  $k_i \in \mathbb{N}$  and order  $\alpha_i > -1$ , see [16, p. 76]. We consider the differential operator

$$(2.1) \quad L_\alpha^H = -\Delta + \|x\|^2 + \sum_{i=1}^n \frac{1}{x_i^2} \left( \alpha_i^2 - \frac{1}{4} \right),$$

here  $\|\cdot\|$  stands for the Euclidean norm. The operator  $L_\alpha^H$  is symmetric and positive in  $L^2(\mathbb{R}_+^n, dx)$  and the Laguerre functions  $\varphi_k^\alpha(x)$  are eigenfunctions of (2.1). Indeed,  $L_\alpha^H \varphi_k^\alpha = (4|k| + 2|\alpha| + 2n) \varphi_k^\alpha$ ; by  $|\alpha|$  and  $|k|$  we denote  $|\alpha| = \alpha_1 + \dots + \alpha_n$  (thus  $|\alpha|$  may be negative) and the length  $|k| = k_1 + \dots + k_n$ . We will refer to  $\varphi_k^\alpha$  as *Laguerre functions of Hermite type*.

For each  $\sigma > 0$ , the fractional integrals for expansions in Laguerre functions of Hermite type are given by

$$(L_\alpha^H)^{-\sigma} f = \sum_{m=0}^{\infty} (4m + 2|\alpha| + 2n)^{-\sigma} P_m f$$

where

$$P_m f = \sum_{|k|=m} a_k^\alpha(f) \varphi_k^\alpha, \quad a_k^\alpha(f) = \int_{\mathbb{R}_+^n} f(x) \varphi_k^\alpha(x) dx.$$

With these notations, our result related to the fractional integrals for the Laguerre expansions of Hermite type is the following one.

**Theorem 2.1.** *Let  $\alpha \in [-1/2, \infty)^n$ . Let  $a \geq 1$ ,  $\sigma > 0$ ,  $1 < p \leq q < \infty$ ,  $1 \leq r \leq \infty$ ,  $t < n/p'$ ,  $s < n/q$ ,  $t + s \geq 0$ .*

(i) *If  $\sigma \geq n/2$ , then there exists a constant depending only on  $\sigma$  and  $\alpha$  such that*

$$\left\| \left( \sum_{j=0}^{\infty} |(L_{\alpha+a_j}^H)^{-\sigma}(f_j)|^r \right)^{1/r} \right\|_{L^q(\mathbb{R}_+^n, \|x\|^{-sq} dx)} \leq C \left\| \left( \sum_{j=0}^{\infty} |f_j|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}_+^n, \|x\|^{tp} dx)}$$

*for all  $f_j \in L^p(\mathbb{R}_+^n, \|x\|^{tp} dx)$ .*

(ii) *If  $\sigma < n/2$ , then the same boundedness holds under the additional condition*

$$\frac{1}{q} \geq \frac{1}{p} - \frac{2\sigma - t - s}{n}.$$

**Remark 2.2.** In the previous theorem, the sequence  $\alpha + aj$  has to be understood as  $(\alpha_1 + aj, \dots, \alpha_n + aj)$ . It will be clear after the proof that this sequence can be changed into  $(\alpha_1 + a_1(j), \dots, \alpha_n + a_n(j))$  where  $\{a_i(j)\}_{j \geq 0}$ , for  $i = 1, \dots, n$ , are positive, increasing and unbounded sequences such that  $a_i(0) = 0$  and  $a_i(1) \geq 1$ .

Let us focus on the other setting that we will analyze in this paper. We consider now the differential operator given by

$$(2.2) \quad L_\alpha = -\Delta + \|x\|^2 - \sum_{i=1}^n \frac{2\alpha_i + 1}{x_i} \frac{\partial}{\partial x_i}.$$

This is a symmetric operator on  $\mathbb{R}_+^n$  equipped with the measure

$$d\mu_\alpha(x) = x_1^{2\alpha_1+1} \cdots x_n^{2\alpha_n+1} dx.$$

The Laguerre functions  $\ell_k^\alpha$  are defined by

$$\ell_k^\alpha(x) = \ell_{k_1}^{\alpha_1} \cdots \ell_{k_n}^{\alpha_n}, \quad x = (x_1, \dots, x_n) \in \mathbb{R}_+^n,$$

where  $\ell_{k_i}^{\alpha_i}$  are the one dimensional Laguerre functions

$$\ell_{k_i}^{\alpha_i}(x_i) = \left( \frac{2\Gamma(k_i + 1)}{\Gamma(k_i + \alpha_i + 1)} \right)^{1/2} L_{k_i}^{\alpha_i}(x_i^2) e^{-x_i^2/2}, \quad x_i > 0, \quad i = 1, \dots, n.$$

The functions  $\ell_k^\alpha$  are eigenfunctions of the differential operator (2.2). Indeed, we have  $L_\alpha \ell_k^\alpha = (4|k| + 2|\alpha| + 2n) \ell_k^\alpha$ . We will refer to the functions  $\ell_k^\alpha$  as *Laguerre functions of convolution type*.

For the expansions of Laguerre functions of convolution type we define the fractional integrals as

$$(L_\alpha)^{-\sigma} f = \sum_{m=0}^{\infty} (4m + 2|\alpha| + 2n)^{-\sigma} \mathcal{P}_m f$$

where  $\sigma > 0$  and

$$\mathcal{P}_m f = \sum_{|k|=m} b_k^\alpha(f) \ell_k^\alpha, \quad b_k^\alpha(f) = \int_{\mathbb{R}_+^n} f(x) \ell_k^\alpha(x) d\mu_\alpha(x).$$

Our result in this case is the following.

**Theorem 2.3.** *Let  $0 < \sigma < \alpha + 1$ ,  $\alpha \geq -1/2$ ,  $a \geq 1$  and  $1 \leq p, q, r \leq \infty$ . Define  $u_j(x) = x^{aj}$ ,  $x \in \mathbb{R}_+$ ,  $j = 0, 1, \dots$ . Assume that  $p$  and  $q$  satisfy*

$$\frac{1}{p} - \frac{\sigma}{\alpha + 1} \leq \frac{1}{q} < \frac{1}{p} - \frac{\sigma}{\alpha + 1},$$

*with exclusion of the cases  $p = 1$  and  $q = \frac{\alpha+1}{\alpha+1-\sigma}$ , and  $p = \frac{\alpha+1}{\sigma}$  and  $q = \infty$ . Then there exists a constant depending only on  $\sigma$  and  $\alpha$  such that*

$$(2.3) \quad \left\| \left( \sum_{j=0}^{\infty} |u_j(L_{\alpha+aj})^{-\sigma} (u_j^{-1} f_j)|^r \right)^{1/r} \right\|_{L^q(\mathbb{R}_+, d\mu_\alpha)} \leq C \left\| \left( \sum_{j=0}^{\infty} |f_j|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}_+, d\mu_\alpha)}$$

for all  $f_j \in L^p(\mathbb{R}_+, d\mu_\alpha)$ .

**Remark 2.4.** In the case of Theorem 2.3, we deduce from our proof that the sequence  $\alpha + aj$  can be changed into  $\alpha + a(j)$  where  $\{a(j)\}_{j \geq 0}$  is a positive, increasing and unbounded sequence such that  $a(j) = 0$  and  $a(1) > 1$ .

### 3. AN APPLICATION OF THEOREM 2.3

As we commented previously in the previous section, the results in Theorem 2.1 and Theorem 2.3 are extensions of some known inequalities for fractional integrals for Laguerre expansions. However, the inequality (2.3) appears in a natural way in the study of fractional integrals related to the harmonic oscillator. Indeed, the eigenfunctions of the harmonic oscillator in  $\mathbb{R}^n$  verify

$$(-\Delta + |x|^2)\phi = E\phi,$$

where  $E$  is the corresponding eigenvalue. There are two complete sets of eigenfunctions for this equation. Using cartesian coordinates, one obtains the functions

$$\phi_k(x) = \prod_{i=1}^n h_{k_i}(x_i), \quad k = (k_1, \dots, k_n) \in \mathbb{N}^n,$$

where  $h_{k_i}(x_i) = (\sqrt{\pi} 2^{k_i} k_i!)^{-1/2} H_{k_i}(x_i) e^{-x_i^2/2}$ , and  $H_j$  denote the Hermite polynomials of degree  $j \in \mathbb{N}$  (see [16, p. 60]). In this case  $E_k = 2|k| + d$ . The system of functions  $\{\phi_k\}_{k \in \mathbb{N}^n}$  is orthonormal and complete in  $L^2(\mathbb{R}^n, dx)$ . The fractional integrals for this system have been treated in [2] and [14]. Vector-valued extensions of the results in both papers for sequences of functions  $\{f_j(x)\}_{j \in \mathbb{N}}$ , with  $x \in \mathbb{R}^n$ , are trivial.

But the situation is completely different if we analyze the eigenfunctions of the harmonic oscillator by using spherical coordinates. Let  $\mathcal{H}_j$  be the space of spherical harmonics of degree  $j$  in  $n$  variables. Let  $\{\mathcal{Y}_{j,\ell}\}_{\ell=1,\dots,\dim \mathcal{H}_j}$  be an orthonormal basis for  $\mathcal{H}_j$  in  $L^2(\mathbb{S}^{n-1}, d\sigma)$ . Then the eigenfunctions of the harmonic oscillator, see [7], are given by

$$\tilde{\phi}_{m,j,\ell}(x) = \left( \frac{2\Gamma(j+1)}{\Gamma(m-j+n/2)} \right)^{1/2} L_j^{n/2-1+m-2j}(r^2) \mathcal{Y}_{m-2j,\ell}(x) e^{-r^2/2}, \quad r^2 = x_1^2 + \dots + x_n^2,$$

where  $m \geq 0$ ,  $j = 0, \dots, [m/2]$ ,  $\ell = 1, \dots, \dim \mathcal{H}_{m-2j}$ , and  $L_j^b$  are Laguerre polynomials of order  $b$  and degree  $j \in \mathbb{N}$ . This system is orthonormal and complete in  $L^2(\mathbb{R}^n, dx)$  and the eigenvalues are  $E_{m,j,\ell} = (n + 2m)$ . Moreover

$$L^2(\mathbb{R}^n, dx) = \bigoplus_{m=0}^{\infty} \mathcal{J}_m$$

with

$$\mathcal{J}_m = \{f \in C^\infty(\mathbb{R}^n) : (-\Delta + |x|^2)f = (n + 2m)f\}.$$

For each  $\sigma > 0$ , we define the fractional integrals for the harmonic oscillator as

$$(-\Delta + |\cdot|)^{-\sigma} f = \sum_{m=0}^{\infty} \frac{1}{(n + 2m)^\sigma} \text{Proj}_{\mathcal{J}_m} f,$$

where

$$\text{Proj}_{\mathcal{J}_m} f = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\ell=1}^{\dim \mathcal{H}_{m-2j}} c_{m,j,\ell}(f) \tilde{\phi}_{m,j,\ell}, \quad c_{m,j,\ell}(f) = \int_{\mathbb{R}^n} \overline{\tilde{\phi}_{m,j,\ell}(y)} f(y) dy.$$

The most appropriate spaces in order to analyze this kind of operators are the mixed norm spaces, defined as

$$L^{p,2}(\mathbb{R}^n, r^{n-1} dr d\sigma) = \{f(x) : \|f\|_{L^{p,2}(\mathbb{R}^n, r^{n-1} dr d\sigma)} < \infty\},$$

where

$$\|f\|_{L^{p,2}(\mathbb{R}^n, r^{n-1} dr d\sigma)} = \left( \int_0^\infty \left( \int_{\mathbb{S}^{n-1}} |f(rx')|^2 d\sigma(x') \right)^{p/2} r^{n-1} dr \right)^{1/p},$$

with the obvious modification in the case  $p = \infty$ . The main characteristic of these spaces is that we consider the  $L^2$ -norm in the angular part and the  $L^p$ -norm in the radial. They are very different from  $L^p(\mathbb{R}^n, dx)$ ; in fact  $L^p(\mathbb{R}^n, dx) \subset L^{p,2}(\mathbb{R}^n, r^{n-1} dr d\sigma)$  for  $p > 2$ ,  $L^2(\mathbb{R}^n, dx) = L^{2,2}(\mathbb{R}^n, r^{n-1} dr d\sigma)$ , and  $L^{p,2}(\mathbb{R}^n, r^{n-1} dr d\sigma) \subset L^p(\mathbb{R}^n, dx)$  for  $p < 2$ . As explained in the introduction, these spaces are the most suitable when spherical harmonics are involved. Indeed, if a function  $f$  on  $\mathbb{R}^n$  is expanded in spherical harmonics,

$$(3.1) \quad f(x) = \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_j} f_{j,\ell}(r) \mathcal{Y}_{j,\ell}\left(\frac{x}{r}\right),$$

where

$$f_{j,\ell}(r) = \int_{\mathbb{S}^{d-1}} f(rx') \overline{\mathcal{Y}_{j,\ell}(x')} d\sigma(x'),$$

we have

$$\|f\|_{L^{p,2}(\mathbb{R}^n, r^{n-1} dr d\sigma)} = \left\| \left( \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_j} |f_{j,\ell}(r)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^+, d\mu_{n/2-1})}.$$

From this and using Theorem 2.3, we can prove the following result.

**Theorem 3.1.** *Let  $n \geq 2$ ,  $0 < \sigma < n/2$ , and  $1 \leq p, q \leq \infty$ . Assume that  $p$  and  $q$  satisfy*

$$\frac{1}{p} - \frac{2\sigma}{n} \leq \frac{1}{q} < \frac{1}{p} - \frac{2\sigma}{n},$$

*with exclusion of the cases  $p = 1$  and  $q = \frac{n}{n-2\sigma}$ , and  $p = \frac{n}{2\sigma}$  and  $q = \infty$ . Then there exists a constant depending only on  $\sigma$  such that*

$$(3.2) \quad \left\| (-\Delta + |\cdot|)^{-\sigma} f \right\|_{L^{q,2}(\mathbb{R}^n, r^{n-1} dr d\sigma)} \leq C \|f\|_{L^{p,2}(\mathbb{R}^n, r^{n-1} dr d\sigma)}$$

*for all  $f \in L^{p,2}(\mathbb{R}^n, r^{n-1} dr d\sigma)$ .*

*Proof.* Consider the decomposition

$$(-\Delta + |\cdot|)^{-\sigma} f = O_1 f + O_2 f$$

where

$$O_1 f = \sum_{k=0}^{\infty} \frac{1}{(n+4k)^\sigma} \text{Proj}_{\mathcal{J}_{2k}} f \quad \text{and} \quad O_2 f = \sum_{k=0}^{\infty} \frac{1}{(n+4k+2)^\sigma} \text{Proj}_{\mathcal{J}_{2k+1}} f.$$

We start analyzing  $O_1$ . After some elementary algebraic manipulations, we have

$$O_1 f = \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_{2j}} \sum_{k=0}^{\infty} \frac{1}{(n+4j+4k)^\sigma} c_{2j+2k,k,\ell}(f) \tilde{\phi}_{2j+2k,k,\ell}.$$

By (3.1), we can deduce the following identity immediately

$$c_{2j+2k,k,\ell}(f) \tilde{\phi}_{2j+2k,k,\ell}(x) = b_k^{n/2-1+2j} ((\cdot)^{-2j} f_{2j,\ell}) r^{2j} \ell_k^{n/2-1+2j}(r) \mathcal{Y}_{2j,\ell}\left(\frac{x}{r}\right),$$

where we used the notation in the previous section for Laguerre expansions of convolution type. Then

$$O_1 f(x) = \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_{2j}} r^{2j} (L_{n/2-1+2j})^{-\sigma} ((\cdot)^{-2j} f_{2j,\ell})(r) \mathcal{Y}_{2j,\ell} \left( \frac{x}{r} \right).$$

In a similar way, we conclude that

$$O_2 f(x) = \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_{2j+1}} r^{2j+1} (L_{n/2+2j})^{-\sigma} ((\cdot)^{-2j-1} f_{2j+1,\ell})(r) \mathcal{Y}_{2j+1,\ell} \left( \frac{x}{r} \right)$$

and

$$(-\Delta + |\cdot|)^{-\sigma} f(x) = \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_j} r^j (L_{n/2-1+j})^{-\sigma} ((\cdot)^{-j} f_{j,\ell})(r) \mathcal{Y}_{j,\ell} \left( \frac{x}{r} \right).$$

So, the inequality (3.2) is equivalent to

$$\begin{aligned} \left\| \left( \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_j} (r^j (L_{n/2-1+j})^{-\sigma} ((\cdot)^{-j} f_{j,\ell})(r))^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^+, d\mu_{n/2-1})} \\ \leq C \left\| \left( \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_j} (f_{j,\ell}(r))^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^+, d\mu_{n/2-1})}, \end{aligned}$$

which is a consequence of Theorem 2.3.  $\square$

#### 4. PROOF OF THEOREM 2.1

The heat semigroup related to  $L_{\alpha}^H$  is initially defined in  $L^2(\mathbb{R}_+^n, d\mu_{\alpha})$  as

$$T_{\alpha,t}^H f = \sum_{m=0}^{\infty} e^{-t(4m+2|\alpha|+2n)} \sum_{|k|=m} \langle f, \varphi_k^{\alpha} \rangle \varphi_k^{\alpha}, \quad t > 0,$$

and by  $\langle f, g \rangle$  we denote  $\int_{\mathbb{R}_+^n} f(x) \overline{g(x)} dx$ . We can write the heat semigroup  $\{T_{\alpha,t}^H\}_{t>0}$  as an integral operator

$$T_{\alpha,t}^H f(x) = \int_{\mathbb{R}_+^n} G_{\alpha,t}^H(x, y) f(y) dy.$$

The Laguerre heat kernel is given by

$$(4.1) \quad G_{\alpha,t}^H(x, y) = \sum_{m=0}^{\infty} e^{-t(4m+2|\alpha|+2n)} \sum_{|k|=m} \varphi_k^{\alpha}(x) \varphi_k^{\alpha}(y).$$

The explicit expression for Laguerre heat kernel is known and it can be found in [16, (4.17.6)]:

$$G_{\alpha,t}^H(x, y) = (\sinh 2t)^{-n} \exp \left( -\frac{1}{2} \coth(2t) (\|x\|^2 + \|y\|^2) \right) \prod_{i=1}^n (x_i y_i)^{1/2} I_{\alpha_i} \left( \frac{x_i y_i}{\sinh 2t} \right),$$

with  $I_{\nu}$  denoting the modified Bessel function of the first kind and order  $\nu$ , see [16, Chapter 5].

We use Schlöfli's integral representation of Poisson's type for modified Bessel function, see [16, (5.10.22)],

$$(4.2) \quad I_{\nu}(z) = z^{\nu} \int_{-1}^1 \exp(-zs) d\Pi_{\nu}(s), \quad |\arg z| < \pi, \quad \nu > -\frac{1}{2},$$

where the measure  $d\Pi_{\nu}(u)$  is given by

$$(4.3) \quad d\Pi_{\nu}(u) = \frac{(1-u^2)^{\nu-1/2} du}{\sqrt{\pi} 2^{\nu} \Gamma(\nu+1/2)}, \quad \nu > -1/2.$$

In the limit case  $\nu = -1/2$ , we put  $\pi_{-1/2} = \frac{1}{2}(\delta_{-1} + \delta_1)$ . Consequently, for  $\alpha \in [-1/2, \infty)^n$ , the kernel can be expressed as

$$G_{\alpha,t}^H(x, y) = (xy)^{\alpha+1/2} (\sinh(2t))^{-n-|\alpha|} \int_{[-1,1]^n} \exp\left(-\frac{1}{2} \coth(2t)(\|x\|^2 + \|y\|^2) - \sum_{i=1}^n \frac{x_i y_i s_i}{\sinh(2t)}\right) d\Pi_\alpha(s),$$

where  $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}^n$ ,  $xy = (x_1 y_1, \dots, x_n y_n)$ ,  $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$  and

$$d\Pi_\alpha(s) = \prod_{i=1}^n \frac{(1-s_i^2)^{\alpha_i-1/2}}{2^{\alpha_i} \sqrt{\pi} \Gamma(\alpha_i + 1/2)} ds_i.$$

Let

$$(4.4) \quad q_\pm = q_\pm(x, y, s) = \|x\|^2 + \|y\|^2 \pm 2 \sum_{i=1}^n x_i y_i s_i.$$

Meda's change of variable

$$(4.5) \quad t = \frac{1}{2} \log \frac{1+\xi}{1-\xi}, \quad \xi \in (0, 1),$$

leads to

$$(4.6) \quad G_{\alpha,t}^H(x, y) = (xy)^{\alpha+1/2} \left(\frac{1-\xi^2}{2\xi}\right)^{n+|\alpha|} \int_{[-1,1]^n} \exp\left(-\frac{1}{4\xi} q_+(x, y, s) - \frac{\xi}{4} q_-(x, y, s)\right) d\Pi_\alpha(s).$$

The following technical lemma can be found in [14, Lemma 2.1].

**Lemma 4.1.** *Let  $a \in \mathbb{R}$  be fixed and  $T > 0$ . Then*

$$\int_0^1 \zeta^{-a} \exp(-T\zeta^{-1}) d\zeta \leq C \exp(-T/2), \quad T \geq 1,$$

and for  $0 < T < 1$

$$\int_0^1 \zeta^{-a} \exp(-T\zeta^{-1}) d\zeta \simeq \begin{cases} 1, & a < 1, \\ \log(2/T), & a = 1, \\ T^{-a+1}, & a > 1. \end{cases}$$

**Proposition 4.2.** *Let  $\alpha \in [-1/2, \infty)^n$ . Then*

$$G_{\alpha,t}^H(x, y) \leq C \left(\frac{1-\xi^2}{\xi}\right)^{n/2} \exp\left(-\frac{\|x-y\|^2}{4\xi} - \frac{\xi\|x+y\|^2}{4}\right),$$

with  $C$  independent of  $\alpha$ .

*Proof.* Let  $q_{\pm,i} = q_{\pm,i}(x_i, y_i, s_i) = x_i^2 + y_i^2 \pm 2x_i y_i s_i$ , for  $i = 1, \dots, n$ . From this identity and (4.4), it follows that  $q_\pm(x, y, s) = \sum_{i=1}^n q_{\pm,i}(x_i, y_i, s_i)$ . Observe that

$$(4.7) \quad \int_{[-1,1]^n} \exp\left(-\frac{q_+}{4\xi} - \frac{\xi q_-}{4}\right) d\Pi_\alpha(s) = \prod_{i=1}^n \int_{-1}^1 \exp\left(-\frac{q_{+,i}}{4\xi} - \frac{\xi q_{-,i}}{4}\right) d\Pi_{\alpha_i}(s_i),$$

so it suffices to deal with the integral in dimension one. The case  $\alpha_i = -1/2$  is elementary, so we obtain the estimate for  $\alpha_i > -1/2$ . We write

$$J := \int_{-1}^1 \exp\left(-\frac{q_{+,i}}{4\xi} - \frac{\xi q_{-,i}}{4}\right) (1-s_i^2)^{\alpha_i-1/2} ds_i.$$

With the change of variable  $s = 2u - 1$  we have

$$J = 4^{\alpha_i} \exp\left(-\frac{(x_i - y_i)^2}{4\xi} - \frac{\xi(x_i + y_i)^2}{4}\right) \int_0^1 \exp\left(-x_i y_i u \left(\frac{1}{\xi} - \xi\right)\right) u^{\alpha_i-1/2} (1-u)^{\alpha_i-1/2} du.$$

It is easy to check that

$$\begin{aligned} \int_{1/2}^1 \exp\left(-x_i y_i u \left(\frac{1}{\xi} - \xi\right)\right) u^{\alpha_i-1/2} (1-u)^{\alpha_i-1/2} du \\ \leq \int_0^{1/2} \exp\left(-x_i y_i u \left(\frac{1}{\xi} - \xi\right)\right) u^{\alpha_i-1/2} (1-u)^{\alpha_i-1/2} du \end{aligned}$$

and then

$$J \leq 4^{\alpha_i+1/2} \exp\left(-\frac{(x_i-y_i)^2}{4\xi} - \frac{\xi(x_i+y_i)^2}{4}\right) \int_0^{1/2} \exp\left(-x_i y_i u \left(\frac{1}{\xi} - \xi\right)\right) u^{\alpha_i-1/2} du.$$

Now, taking  $x_i y_i u \left(\frac{1}{\xi} - \xi\right) = z$ , we get

$$\begin{aligned} J &= \frac{4^{\alpha_i}}{(x_i y_i)^{\alpha_i+1/2}} \left(\frac{\xi}{1-\xi^2}\right)^{\alpha_i+1/2} \exp\left(-\frac{(x_i-y_i)^2}{4\xi} - \frac{\xi(x_i+y_i)^2}{4}\right) \int_0^{x_i y_i (1-\xi^2)/(2\xi)} e^{-z} z^{\alpha_i-1/2} dz \\ &\leq \frac{4^{\alpha_i}}{(x_i y_i)^{\alpha_i+1/2}} \left(\frac{\xi}{1-\xi^2}\right)^{\alpha_i+1/2} \exp\left(-\frac{(x_i-y_i)^2}{4\xi} - \frac{\xi(x_i+y_i)^2}{4}\right) \Gamma(\alpha_i+1/2). \end{aligned}$$

From (4.3), (4.6) and (4.7), the estimate in the proposition follows.  $\square$

For each  $\sigma > 0$ , we define the potential integral operator

$$(4.8) \quad \mathcal{I}_{\alpha,\sigma}^H f(x) = \int_{\mathbb{R}_+^n} \mathcal{H}_{\alpha,\sigma}^H(x,y) f(y) dy, \quad x \in \mathbb{R}_+^n,$$

where

$$(4.9) \quad \mathcal{H}_{\alpha,\sigma}^H(x,y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty G_{\alpha,t}^H(x,y) t^{\sigma-1} dt, \quad x, y \in \mathbb{R}_+^n$$

is the potential kernel.

Define the auxiliary convolution kernel  $K^\sigma(x)$ ,  $x \in \mathbb{R}^n \setminus \{0\}$ , by

$$(4.10) \quad K_\sigma(x) = \exp(-c\|x\|^2), \quad \|x\| \geq 1,$$

and for  $\|x\| < 1$ ,

$$(4.11) \quad K_\sigma(x) = \begin{cases} 1, & \sigma > n/2, \\ \log\left(\frac{c}{\|x\|}\right), & \sigma = n/2, \\ \frac{1}{\|x\|^{n-2\sigma}}, & \sigma < n/2. \end{cases}$$

**Proposition 4.3.** *Let  $\sigma > 0$  and  $\alpha \in [-1/2, \infty)^n$ . Then*

$$\mathcal{H}_{\alpha,\sigma}^H(x,y) \leq C_\sigma K_\sigma(x-y),$$

with  $C_\sigma$  independent of  $\alpha$ .

*Proof.* By (4.9), Meda's change of variable, (4.6), and Proposition 4.2, we have

$$\begin{aligned} \mathcal{H}_{\alpha,\sigma}^H(x,y) &\leq \frac{C}{2^{\sigma-1}} \int_0^1 \left(\log \frac{1+\xi}{1-\xi}\right)^{\sigma-1} (1-\xi^2)^{n/2-1} \xi^{-n/2} \exp\left(-\frac{1}{4\xi} \|x-y\|^2\right) d\xi \\ &= \int_0^{1/2} + \int_{1/2}^1 := I_1 + I_2. \end{aligned}$$

Concerning  $I_1$ , observe that there exists  $C$  such that  $\log \frac{1+\xi}{1-\xi} < C\xi$ , for  $\xi \in (0, 1/2)$ . Then, we apply Lemma 4.1 with  $a = -\sigma + 1 + n/2$  and  $T = \|x-y\|^2/4$ , and we obtain

$$I_1 \leq C_\sigma \int_0^{1/2} \xi^{\sigma-1-n/2} \exp\left(-\frac{1}{4\xi} \|x-y\|^2\right) d\xi \leq C_\sigma \begin{cases} 1, & \sigma > n/2, \\ \log\left(\frac{c}{\|x-y\|}\right), & \sigma = n/2, \\ \frac{1}{\|x-y\|^{n-2\sigma}}, & \sigma < n/2. \end{cases}$$



It is easy to see that, if  $\|x - y\| \geq 1$ , then  $I_1 \leq C_\sigma \exp(-c\|x - y\|^2)$ . Indeed, in this case,

$$I_1 \leq C_\sigma \exp(-c\|x - y\|^2) \int_0^{1/2} \xi^{\sigma-1-n/2} \exp\left(-\frac{1}{8\xi}\|x - y\|^2\right) d\xi$$

and applying Lemma 4.1, we get the estimate. Now we deal with  $I_2$ . Using that  $\xi^{-n/2} \sim 1$ , for  $\xi \in (1/2, 1)$ , and reverting Meda's change of variable yield

$$\begin{aligned} I_2 &\leq C_\sigma \exp(-c\|x - y\|^2) \int_{1/2}^1 \left(\log\left(\frac{1+\xi}{1-\xi}\right)\right)^{\sigma-1} (1-\xi^2)^{n/2-1} d\xi \\ &\leq C_\sigma \exp(-c\|x - y\|^2) \int_{\log 3}^\infty w^{\sigma-1} \exp(-wn/2) dw \leq C_\sigma \exp(-c\|x - y\|^2). \end{aligned}$$

□

On the other hand, we will use the following result about vector-valued extensions of bounded operators. This result is a version of [9, Ch. 5, Theorem 1.12] in setting of  $\ell^r$  spaces.

**Lemma 4.4.** *Consider  $L^p = L^p(X, m)$ , where  $(X, m)$  is a  $\sigma$ -finite measure space. Let  $T : L^p \rightarrow L^q$  be a bounded linear operator which is positive (i.e.  $g(x) \geq 0$  implies  $Tg(x) \geq 0$ ),  $1 \leq p, q \leq \infty$ , with norm  $\|T\|$ . Then  $T$  has an  $\ell^r$ -valued extension for  $1 \leq r \leq \infty$  and*

$$\left\| \left( \sum_j |Tf_j|^r \right)^{1/r} \right\|_{L^q} \leq \|T\| \left\| \left( \sum_j |f_j|^r \right)^{1/r} \right\|_{L^p}, \quad f_j \in L^p.$$

*Proof of Theorem 2.1.* First we have to prove that  $(L_\alpha^H)^{-\sigma} = \mathcal{I}_{\alpha, \sigma}^H$  as operators on  $L^2(\mathbb{R}_+^n, dx)$ . This is obtained by showing that both operators, being bounded in  $L^2(\mathbb{R}_+^n, dx)$ , coincide on the dense in  $L^2(\mathbb{R}_+^n, dx)$  linear span of the functions  $\varphi_k^\alpha$ . Indeed, in order to check  $(L_\alpha^H)^{-\sigma} \varphi_k^\alpha = \mathcal{I}_{\alpha, \sigma}^H \varphi_k^\alpha$ , we write

$$\begin{aligned} \int_{\mathbb{R}_+^n} \mathcal{H}_{\alpha, \sigma}^H(x, y) \varphi_k^\alpha(y) dy &= \int_{\mathbb{R}_+^n} \int_0^\infty G_{\alpha, t}^H(x, y) t^{\sigma-1} dt \varphi_k^\alpha(y) dy \\ &= \varphi_k^\alpha(x) \int_0^\infty e^{-t(4|k|+2|\alpha|+2n)} t^{\sigma-1} dt = \Gamma(\sigma) (L_\alpha^H)^{-\sigma} \varphi_k^\alpha(x). \end{aligned}$$

Application of Fubini's theorem in the second identity was possibly since  $\mathcal{H}_{\alpha, \sigma}^H(x, \cdot) \leq K_\sigma(x - \cdot) \in L^1(\mathbb{R}^n, dx)$ , for each  $x \in \mathbb{R}_+^n$ , and  $\varphi_k^\alpha \in L^\infty(\mathbb{R}_+^n, dx)$ .

Now we observe that, by Theorem 4.3, there exists a constant  $C_\sigma$  depending only on  $\sigma$  such that for a nonnegative function  $f$ ,

$$\mathcal{I}_{\alpha+aj, \sigma}^H(f)(x) \leq C_\sigma \int_{\mathbb{R}_+^n} K_\sigma(x - y) f(y) dy.$$

By [14, Theorem 2.5] and Lemma 4.4 (note that  $K_\sigma(x - y)$  is positive), there exists a constant  $C$  depending only on  $\sigma, s$  and  $t$  such that

$$\left\| \left( \sum_{j=0}^\infty \left| \int_{\mathbb{R}_+^n} K_\sigma(x - y) f(y) dy \right|^r \right)^{1/r} \right\|_{L^q(\mathbb{R}_+^n, \|x\|^{-sq} dx)} \leq C \left\| \left( \sum_{j=0}^\infty |f_j|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}_+^n, \|x\|^{tp} dx)},$$

for  $f_j \in L^p(\mathbb{R}_+^n, \|x\|^{tp} dx)$ .

Therefore,

$$\left\| \left( \sum_{j=0}^\infty |\mathcal{I}_{\alpha+aj, \sigma}^H(f_j)|^r \right)^{1/r} \right\|_{L^q(\mathbb{R}_+^n, \|x\|^{-sq} dx)} \leq C \left\| \left( \sum_{j=0}^\infty |f_j|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}_+^n, \|x\|^{tp} dx)},$$

and the proof is complete. □

## 5. PROOF OF THEOREM 2.3

Recall the differential operator  $L_\alpha$  given in (2.2). The heat semigroup related to  $L_\alpha$  is initially defined in  $L^2(\mathbb{R}_+^n, d\mu_\alpha)$  as

$$T_{\alpha,t}f = \sum_{m=0}^{\infty} e^{-t(4m+2|\alpha|+2n)} \sum_{|k|=m} \langle f, \ell_k^\alpha \rangle_{d\mu_\alpha} \ell_k^\alpha, \quad t > 0,$$

and by  $\langle f, g \rangle_{d\mu_\alpha}$  we denote  $\int_{\mathbb{R}_+^n} f(x) \overline{g(x)} d\mu_\alpha(x)$ . We can write the heat semigroup  $\{T_{\alpha,t}\}_{t>0}$  as an integral operator

$$T_{\alpha,t}f(x) = \int_{\mathbb{R}_+^n} G_{\alpha,t}(x, y) f(y) d\mu_\alpha(y).$$

The Laguerre heat kernel in this case is given by

$$(5.1) \quad G_{\alpha,t}(x, y) = \sum_{m=0}^{\infty} e^{-t(4m+2|\alpha|+2n)} \sum_{|k|=m} \ell_k^\alpha(x) \ell_k^\alpha(y).$$

Observe that

$$(5.2) \quad G_{\alpha,t}(x, y) = G_{\alpha,t}^H(x, y)(xy)^{-\alpha-1/2}.$$

We define the potential operator

$$(5.3) \quad \mathcal{I}_{\alpha,\sigma}f(x) = \int_{\mathbb{R}_+^n} \mathcal{H}_{\alpha,\sigma}(x, y) f(y) d\mu_\alpha(y), \quad x \in \mathbb{R}_+^n,$$

where

$$(5.4) \quad \mathcal{H}_{\alpha,\sigma}(x, y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty G_{\alpha,t}(x, y) t^{\sigma-1} dt, \quad x, y \in \mathbb{R}_+^n$$

is the potential kernel. Due to (5.2),

$$(5.5) \quad \mathcal{H}_{\alpha,\sigma}(x, y) = \mathcal{H}_{\alpha,\sigma}^H(x, y)(xy)^{-\alpha-1/2}.$$

For,  $n = 1$ , by using the estimates in the previous section for  $\mathcal{H}_{\alpha,\sigma}^H$  and the relation (5.5), we obtain that

$$\mathcal{H}_{\alpha+aj,\sigma}(x, y) \leq C_\sigma K_\sigma(x - y)(xy)^{-\alpha-aj-1/2}.$$

If one uses this estimate to attain  $L^p - L^q$  inequalities, then it turns out that we need an extra restriction on the parameters. In fact, the condition  $\frac{4(\alpha+1)}{2\alpha+3} < p \leq q < \frac{4(\alpha+1)}{2\alpha+1}$  appears as a consequence of the presence of the factor  $(xy)^{-\alpha-1/2}$  and the measure  $d\mu_\alpha$ , in the estimate of  $\mathcal{H}_{\alpha+aj,\sigma}(x, y)$ . So, in order to take out such restriction on  $p$  and  $q$ , one has to get suitable estimates for the kernel. In this way, the proof of Theorem 2.3 is based on the estimates collected in Propositions 5.1 and 5.5.

**Proposition 5.1.** *Let  $\alpha \geq -1/2$ ,  $a \geq 1$ ,  $j \in \mathbb{N} \cup \{0\}$ , and  $0 < \sigma < \alpha + 1$ . Then*

$$\mathcal{H}_{\alpha+aj,\sigma}(x, y) \leq C(xy)^{-aj} \mathcal{K}_{\alpha,\sigma}(x, y),$$

where  $C$  depends on  $\alpha$  and  $\sigma$ , but not on  $j$ , and

$$(5.6) \quad \mathcal{K}_{\alpha,\sigma}(x, y) = \begin{cases} \frac{1}{|x-y|^{1-2\sigma}(x+y)^{2\alpha+1}}, & |x-y| < 1, \\ \frac{e^{-c(x-y)^2}}{(x+y)^{2\alpha+1}}, & |x-y| \geq 1, \end{cases}$$

for some constant  $c > 0$ .

The two following lemmas will provide us the main tools to prove the previous proposition in the case  $|x-y| < 1$ .

**Lemma 5.2.** *Let  $c > -1$  and  $\ell$  be such that  $0 < \sigma < c + \ell$ . Then*

$$\int_0^1 \left( \log \left( \frac{1+\xi}{1-\xi} \right) \right)^{\sigma-1} (1-\xi^2)^c \xi^{-c-\ell} \exp \left( -\frac{q_+}{4\xi} \right) d\xi \leq C \frac{4^c \Gamma(c+\ell-\sigma)}{q_+^{c+\ell-\sigma}},$$

where  $C$  is independent of  $c$ .

**Lemma 5.3.** *Let  $\alpha \geq -1/2$ ,  $\lambda > 0$ ,  $b \geq 0$ , and  $0 < B < A$ . Then*

$$\int_0^1 \frac{(1-s)^{\alpha+b-1/2}}{(A-Bs)^{\alpha+b+\lambda+1/2}} ds \leq C(b) \frac{1}{A^{\alpha+1/2}} \frac{1}{B^b} \frac{1}{(A-B)^\lambda},$$

where

$$C(b) = \begin{cases} \frac{\Gamma(b)\Gamma(\lambda)}{\Gamma(b+\lambda)}, & b > 0, \\ C_\alpha, & b = 0. \end{cases}$$

In the case  $|x-y| \geq 1$ , the result will follow from an estimate of the heat kernel similar to the one in Proposition 4.2.

**Lemma 5.4.** *Let  $\alpha \geq -1/2$ ,  $a \geq 1$ , and  $j \in \mathbb{N} \cup \{0\}$ . Then*

$$G_{\alpha+aj,t}(x,y) \leq C(j) \exp \left( -\frac{(x-y)^2}{8\xi} \right) \frac{(1-\xi^2)^{\alpha+1}}{\xi^{\alpha+1}} |x^2 - y^2|^{-(2\alpha+1)} (xy)^{-aj},$$

where

$$C(j) = C \begin{cases} \frac{\Gamma(aj)}{\Gamma(\alpha+aj+1/2)}, & j \geq 1, \\ 1, & j = 0, \end{cases}$$

and  $C$  is independent of  $j$ .

The previous lemmas are rather technical and their proofs will be given in the last section.

*Proof of Proposition 5.1.* We analyze the case  $j \geq 1$ ; the case  $j = 0$  is similar. For  $|x-y| < 1$ , by (5.4), (5.2), Meda's changes of variable, (4.6), Lemma 5.2 with  $c = \alpha + aj$  and  $\ell = 1$ , and the change of variable  $s = 1 - 2u$ , we have

$$\begin{aligned} \mathcal{H}_{\alpha+aj,\sigma}(x,y) &\leq C \frac{\Gamma(\alpha+aj+1-\sigma)}{\Gamma(\alpha+aj+1/2)} \int_{-1}^1 \frac{(1-s^2)^{\alpha+aj-1/2}}{q_+^{\alpha+aj+1-\sigma}} ds \\ &= C 4^{\alpha+aj} \frac{\Gamma(\alpha+aj+1-\sigma)}{\Gamma(\alpha+aj+1/2)} \int_0^1 \frac{(1-u)^{\alpha+aj-1/2} u^{\alpha+aj-1/2}}{((x+y)^2 - 4xyu)^{\alpha+aj+1-\sigma}} du \\ &\leq C 4^{\alpha+aj} \frac{\Gamma(\alpha+aj+1-\sigma)}{\Gamma(\alpha+aj+1/2)} \int_0^1 \frac{(1-u)^{\alpha+aj-1/2}}{((x+y)^2 - 4xyu)^{\alpha+aj+1-\sigma}} du. \end{aligned}$$

Finally, we conclude by using Lemma 5.3 with  $b = aj$ ,  $\lambda = 1/2 - \sigma$ ,  $A = (x+y)^2$ , and  $B = 4xy$ . Indeed,

$$\begin{aligned} \mathcal{H}_{\alpha+aj,\sigma}(x,y) &\leq C 4^{\alpha+aj} \frac{\Gamma(\alpha+aj+1-\sigma)}{\Gamma(\alpha+aj+1/2)} \frac{\Gamma(aj)}{\Gamma(aj+1/2-\sigma)} \frac{1}{(x+y)^{2\alpha+1}} \frac{1}{(4xy)^{aj}} \frac{1}{|x-y|^{1-2\sigma}} \\ &\leq C \frac{1}{(x+y)^{2\alpha+1}} \frac{1}{(xy)^{aj}} \frac{1}{|x-y|^{1-2\sigma}}. \end{aligned}$$

To bound the kernel in the case  $|x-y| \geq 1$ , we use (5.4), Meda's changes of variable and Lemma 5.4 to obtain that

$$\begin{aligned} \mathcal{H}_{\alpha+aj,\sigma}(x,y) &\leq C(x+y)^{-(2\alpha+1)} (xy)^{-aj} \exp \left( -\frac{(x-y)^2}{16} \right) \\ &\quad \times \int_0^1 \left( \log \left( \frac{1+\xi}{1-\xi} \right) \right)^{\sigma-1} (1-\xi^2)^\alpha \xi^{-\alpha-1} \exp \left( -\frac{(x-y)^2}{16\xi} \right) d\xi. \end{aligned}$$

The last integral can be controlled by a constant applying Lemma 5.2 with  $c = \alpha$  and  $\ell = 1$ , and the condition  $|x - y| \geq 1$ . Then

$$\mathcal{H}_{\alpha+aj,\sigma}(x, y) \leq C(x+y)^{-(2\alpha+1)}(xy)^{-aj} \exp\left(-\frac{(x-y)^2}{16\xi}\right)$$

and the proof is finished.  $\square$

The next auxiliary result will be used in the proof of Theorem 2.3.

**Proposition 5.5.** *Let  $\alpha \geq -1/2$ ,  $a \geq 1$ ,  $j \in \mathbb{N} \cup \{0\}$ , and  $0 < \sigma < \alpha + 1$ . Then,*

$$(5.7) \quad u_j(x)(L_{\alpha+aj})^{-\sigma}(u_j^{-1}f)(x) \leq C \int_0^\infty f(y) \bar{\mathcal{K}}_{\alpha,\sigma}(x, y) d\mu_\alpha(y),$$

where  $C$  is independent of  $j$  and

$$\bar{\mathcal{K}}_{\alpha,\sigma}(x, y) = (xy)^{-\alpha-1/2} \int_0^1 \left( \log \left( \frac{1+\xi}{1-\xi} \right) \right)^{\sigma-1} \xi^{-1/2} (1-\xi^2)^{-1/2} \exp\left(-\frac{(x-y)^2}{4\xi} - \frac{\xi(x+y)^2}{4}\right) d\xi.$$

Moreover, for  $x > 2$ ,

$$(5.8) \quad x^{2\sigma} \int_0^\infty \bar{\mathcal{K}}_{\alpha,\sigma}(x, y) d\mu_\alpha(y) \leq C.$$

Another tool we need is a result about fractional integrals in spaces of homogeneous type.

Let  $(X, \mu, |\cdot|)$  be a space of homogeneous type. Given a weight  $w$  and a ball  $B$ ,  $w(B)$  will denote  $\int_B w d\mu$  and  $\mu(B)$  the measure of the ball. We can define the fractional integral operator as

$$I_\gamma f(x) = \int_X \frac{f(y)}{(\mu(B(x, |x-y|)))^{1-\gamma}} d\mu(y),$$

where  $\mu(B(x, |x-y|))$  denotes the measure of the ball of center  $x$  and radius  $|x-y|$ . We will use the following result in [1] concerning  $L^q - L^p$  mapping properties for  $I_\gamma$ .

**Theorem 5.6** (Theorem 1.6 and Remark 1.10 in [1]). *Let  $0 \leq \gamma < 1$  and  $1 < p \leq q < \infty$ . Let  $(w, v)$  be a pair of weights with  $u = v^{-1/(p-1)} \in A_\infty$ . Then*

$$\|I_\gamma f\|_{L^q(X, w d\mu)} \leq C \|f\|_{L^p(X, v d\mu)}$$

if and only if

$$\frac{w(B)^{p/q} u(B)^{p-1}}{\mu(B)^{(1-\gamma)p}} \leq C, \quad \text{for every ball } B \subset X.$$

The proof of Theorem 2.3 will be obtained from the following result and the Riesz-Thorin interpolation theorem.

**Theorem 5.7.** *Let  $\alpha \geq -1/2$ ,  $0 < \sigma < \alpha + 1$ ,  $a \geq 1$ , and  $1 \leq p, q, r \leq \infty$ . Then:*

a) *If  $1 - \frac{\sigma}{\alpha+1} < \frac{1}{q} \leq 1$ , there exists a constant  $C$  such that*

$$\left\| \left( \sum_{j=0}^\infty |u_j(x)(L_{\alpha+aj})^{-\sigma}(u_j^{-1}f_j)|^r \right)^{1/r} \right\|_{L^q(\mathbb{R}_+, d\mu_\alpha)} \leq C \left\| \left( \sum_{j=0}^\infty |f_j|^r \right)^{1/r} \right\|_{L^1(\mathbb{R}_+, d\mu_\alpha)}.$$

b) *If  $\frac{1}{p} < \frac{\sigma}{\alpha+1}$ , there exists a constant  $C$  such that*

$$\left\| \left( \sum_{j=0}^\infty |u_j(x)(L_{\alpha+aj})^{-\sigma}(u_j^{-1}f_j)|^r \right)^{1/r} \right\|_{L^\infty(\mathbb{R}_+, d\mu_\alpha)} \leq C \left\| \left( \sum_{j=0}^\infty |f_j|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}_+, d\mu_\alpha)}.$$

c) *If  $\frac{1}{q} < \frac{\sigma}{\alpha+1}$ , there exists a constant  $C$  such that*

$$\left\| \left( \sum_{j=0}^\infty |u_j(x)(L_{\alpha+aj})^{-\sigma}(u_j^{-1}f_j)|^r \right)^{1/r} \right\|_{L^q(\mathbb{R}_+, d\mu_\alpha)} \leq C \left\| \left( \sum_{j=0}^\infty |f_j|^r \right)^{1/r} \right\|_{L^\infty(\mathbb{R}_+, d\mu_\alpha)}.$$

d) If  $1 - \frac{\sigma}{\alpha+1} < \frac{1}{p} \leq 1$ , there exists a constant  $C$  such that

$$\left\| \left( \sum_{j=0}^{\infty} |u_j(x)(L_{\alpha+aj})^{-\sigma}(u_j^{-1}f_j)|^r \right)^{1/r} \right\|_{L^1(\mathbb{R}_+, d\mu_\alpha)} \leq C \left\| \left( \sum_{j=0}^{\infty} |f_j|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}_+, d\mu_\alpha)}.$$

e) If  $p > 1$ ,  $q < \infty$  and  $\frac{1}{p} - \frac{\sigma}{\alpha+1} = \frac{1}{q}$ , there exists a constant  $C$  such that

$$\left\| \left( \sum_{j=0}^{\infty} |u_j(x)(L_{\alpha+aj})^{-\sigma}(u_j^{-1}f_j)|^r \right)^{1/r} \right\|_{L^q(\mathbb{R}_+, d\mu_\alpha)} \leq C \left\| \left( \sum_{j=0}^{\infty} |f_j|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}_+, d\mu_\alpha)}.$$

*Proof.* As in the proof of Theorem 2.1, it can be checked easily that  $(L_\alpha)^{-\sigma} = \mathcal{I}_{\alpha, \sigma}$  so we omit the details.

By (5.3) and Proposition 5.1, it is clear that

$$u_j(x)(L_{\alpha+aj})^{-\sigma}(u_j^{-1}f_j) \leq C \int_0^\infty f_j(y) \mathcal{K}_{\alpha, \sigma}(x, y) d\mu_\alpha(y),$$

where  $\mathcal{K}_{\alpha, \sigma}$  is as in (5.6). Then the inequality in a) will be deduced from Lemma 4.4 and the estimate

$$\left\| \int_0^\infty f(y) \mathcal{K}_{\alpha, \sigma}(x, y) d\mu_\alpha(y) \right\|_{L^q(d\mu_\alpha)} \leq C \|f\|_{L^1(d\mu_\alpha)}.$$

Applying Minkowski inequality, the previous inequality is a consequence of the estimate

$$(5.9) \quad \|\mathcal{K}_{\alpha, \sigma}(\cdot, y)\|_{L^q(d\mu_\alpha)} \leq C.$$

To prove the previous bound, we consider the cases  $|x - y| < 1$  and  $|x - y| \geq 1$ . For  $|x - y| < 1$  and  $y \geq 1$ , we have

$$\int_{|x-y|<1} (\mathcal{K}_{\alpha, \sigma}(x, y))^q d\mu_\alpha(x) \leq C y^{(2\alpha+2)(1-q)+2q\sigma} \int_{|t-1|<1/y} |1-t|^{-q(1-2\sigma)} dt \sim y^{(2\alpha+1)(1-q)} \leq C.$$

When  $y < 1$ , it is verified that

$$\begin{aligned} \int_{|x-y|<1} (\mathcal{K}_{\alpha, \sigma}(x, y))^q d\mu_\alpha(x) &\leq C y^{(2\alpha+2)(1-q)+2q\sigma} \int_0^{1+1/y} \frac{t^{2\alpha+1}}{|1-t|^{q(1-2\sigma)}(1+t)^{q(2\alpha+1)}} dt \\ &\sim 1 + y^{(2\alpha+2)(1-q)+2q\sigma} \leq C, \end{aligned}$$

where in the last step we used that  $1 - \frac{\sigma}{\alpha+1} < \frac{1}{q}$ . For  $|x - y| \geq 1$ , we have

$$\int_{|x-y|\geq 1} (\mathcal{K}_{\alpha, \sigma}(x, y))^q d\mu_\alpha(x) \leq C \int_{|x-y|\geq 1} \frac{e^{-c(x-y)^2}}{(x+y)^{q(2\alpha+1)}} d\mu_\alpha(x).$$

In this region, it holds  $x^{2\alpha+1}(x+y)^{-q(2\alpha+1)} \leq C$ , then

$$\int_{|x-y|\geq 1} (\mathcal{K}_{\alpha, \sigma}(x, y))^q d\mu_\alpha(x) \leq C \int_{|x-y|\geq 1} e^{-c(x-y)^2} dx \leq C,$$

and the proof of (5.9) is completed.

To prove b), using an argument analogous to a), it is enough to prove that

$$\left\| \int_0^\infty f(y) \mathcal{K}_{\alpha, \sigma}(x, y) d\mu_\alpha(y) \right\|_{L^\infty(d\mu_\alpha)} \leq C \|f\|_{L^p(d\mu_\alpha)}.$$

Now, by Hölder inequality, the result will follow from the estimate (5.9) and the symmetry of the kernel  $\mathcal{K}_{\alpha, \sigma}$ , using the condition  $\frac{1}{p} < \frac{\sigma}{\alpha+1}$ .

To prove c), we start considering  $x \in (0, 2)$  and  $x \geq 2$ . In the first case, by Proposition 5.1 and Lemma 4.4, the inequality is reduced to prove

$$\left\| \chi_{(0,2)}(x) \int_0^\infty f(y) \mathcal{K}_{\alpha, \sigma}(x, y) d\mu_\alpha(y) \right\|_{L^q(d\mu_\alpha)} \leq C \|f\|_{L^\infty(d\mu_\alpha)}.$$

Now, taking into account that

$$\int_0^\infty f(y) \mathcal{K}_{\alpha,\sigma}(x, y) d\mu_\alpha(y) \leq \|f\|_{L^\infty(d\mu_\alpha)} \int_0^\infty \mathcal{K}_{\alpha,\sigma}(x, y) d\mu_\alpha(y),$$

we will conclude by showing that

$$\left\| \chi_{(0,2)}(x) \int_0^\infty \mathcal{K}_{\alpha,\sigma}(x, y) d\mu_\alpha(y) \right\|_{L^q(d\mu_\alpha)} \leq C,$$

but this is true by (5.9) with  $q = 1$ .

When  $x \geq 2$ , by (5.7) and Lemma 4.4, it will be enough to prove that

$$\left\| \chi_{[2,\infty)}(x) \int_0^\infty f(y) \bar{\mathcal{K}}_{\alpha,\sigma}(x, y) d\mu_\alpha(y) \right\|_{L^q(d\mu_\alpha)} \leq C \|f\|_{L^\infty(d\mu_\alpha)}.$$

But, applying (5.8), we obtain that

$$\begin{aligned} \int_2^\infty \left( \int_0^\infty f(y) \bar{\mathcal{K}}_{\alpha,\sigma}(x, y) d\mu_\alpha(y) \right)^q d\mu_\alpha(x) &\leq \|f\|_{L^\infty(d\mu_\alpha)}^q \int_2^\infty \left( \int_0^\infty \bar{\mathcal{K}}_{\alpha,\sigma}(x, y) d\mu_\alpha(y) \right)^q d\mu_\alpha(x) \\ &\leq C \|f\|_{L^\infty(d\mu_\alpha)}^q \int_2^\infty x^{2\alpha+1-2\sigma q} dx \leq C \|f\|_{L^\infty(d\mu_\alpha)}^q, \end{aligned}$$

where in the last step we have used the restriction  $\frac{1}{q} < \frac{\sigma}{\alpha+1}$ .

Now we proceed with the proof of d). We distinguish between  $y \in (0, 2)$  and  $y \geq 2$ . In the first case, we observe that, by Proposition 5.1 and Lemma 4.4, the inequality is reduced to prove

$$\left\| \int_0^2 f(y) \mathcal{K}_{\alpha,\sigma}(x, y) d\mu_\alpha(y) \right\|_{L^1(d\mu_\alpha)} \leq C \|f\|_{L^p(d\mu_\alpha)}.$$

By Fubini's theorem and Hölder inequality

$$\int_0^\infty \int_0^2 f(y) \mathcal{K}_{\alpha,\sigma}(x, y) d\mu_\alpha(y) d\mu_\alpha(x) \leq \|f\|_{L^p(d\mu_\alpha)} \int_0^2 \|\mathcal{K}_{\alpha,\sigma}(\cdot, y)\|_{L^{p'}(d\mu_\alpha)} d\mu_\alpha(y) \leq C \|f\|_{L^p(d\mu_\alpha)},$$

where in the last step we have used (5.9).

In the case  $y \geq 2$ , by (5.7) and Lemma 4.4, it will be enough to prove that

$$\left\| \int_2^\infty f(y) \bar{\mathcal{K}}_{\alpha,\sigma}(x, y) d\mu_\alpha(y) \right\|_{L^1(d\mu_\alpha)} \leq C \|f\|_{L^p(d\mu_\alpha)}.$$

Applying Fubini's theorem, Hölder inequality and (5.8), we obtain that

$$\begin{aligned} \int_0^\infty \int_2^\infty f(y) \bar{\mathcal{K}}_{\alpha,\sigma}(x, y) d\mu_\alpha(y) d\mu_\alpha(x) &\leq \|f\|_{L^p(d\mu_\alpha)} \left\| \chi_{(2,\infty)}(y) \int_0^\infty \bar{\mathcal{K}}_{\alpha,\sigma}(x, y) d\mu_\alpha(x) \right\|_{L^{p'}(d\mu_\alpha)} \\ &\leq C \|f\|_{L^p(d\mu_\alpha)} \left\| \chi_{(2,\infty)}(y) y^{-2\sigma} \right\|_{L^{p'}(d\mu_\alpha)} \leq C \|f\|_{L^p(d\mu_\alpha)}, \end{aligned}$$

where in the last step, we used that  $1 - \frac{\sigma}{\alpha+1} < \frac{1}{p} \leq 1$ .

In the end, to prove e) we observe that  $\mu_\alpha(B(x, |x-y|)) \sim |x-y|(x+y)^{2\alpha+1}$ . Then, by Proposition 5.1

$$\begin{aligned} u_j(x) (L_{\alpha+aj})^{-\sigma} (u_j^{-1} f_j)(x) &\leq C \int_0^\infty \frac{f(y)}{|x-y|^{1-2\sigma} (x+y)^{2\alpha+1}} d\mu_\alpha(y) \\ &\leq C \int_0^\infty \frac{f(y)}{(\mu_\alpha(B(x, |x-y|)))^{(1-\sigma/(\alpha+1))}} d\mu_\alpha(y) = I_{\sigma/(\alpha+1)} f(x). \end{aligned}$$

So, the result is an immediate consequence of Theorem 5.6 and Lemma 4.4.  $\square$

## 6. PROOFS OF TECHNICAL RESULTS

*Proof of Lemma 5.2.* First, observe that  $\log\left(\frac{1+\xi}{1-\xi}\right) \sim \xi$ , for  $0 < \xi \leq 1/2$ , and  $\log\left(\frac{1+\xi}{1-\xi}\right) \sim -\log(1-\xi^2)$ , for  $1/2 < \xi < 1$ . Then, denoting by  $J$  the integral to be estimated, we have

$$J \leq C \int_0^{1/2} \xi^{\sigma-c-\ell-1} \exp\left(-\frac{q_+}{4\xi}\right) d\xi + C \int_{1/2}^1 (-\log(1-\xi^2))^{\sigma-1} (1-\xi^2)^c \xi^{-c-\ell} \exp\left(-\frac{q_+}{4\xi}\right) d\xi.$$

Let  $J_1$  and  $J_2$  be, respectively, the two integrals in the right hand side of the previous inequality. Now, for  $J_1$  the change of variable  $s = \frac{q_+}{4\xi}$  produces the required bound. Indeed,

$$J_1 = \frac{4^{c+\ell-\sigma}}{q_+^{c+\ell-\sigma}} \int_{\frac{q_+}{4}}^{\infty} e^{-s} s^{c+\ell-\sigma-1} ds \leq C \frac{4^c \Gamma(c+\ell-\sigma)}{q_+^{c+\ell-\sigma}}.$$

In order to control  $J_2$ , we start by using the estimate

$$t^\gamma e^{-t} \leq \gamma^\gamma e^{-\gamma}, \quad t, \gamma > 0,$$

to deduce that

$$\xi^{-c-\ell} \exp\left(-\frac{q_+}{4\xi}\right) \leq \frac{4^{c+\ell-\sigma}}{q_+^{c+\ell-\sigma}} \xi^{-\sigma} (c+\ell-\sigma)^{c+\ell-\sigma} e^{-(c+\ell-\sigma)}.$$

Then,

$$\begin{aligned} J_2 &\leq \frac{4^{c+\ell-\sigma}}{q_+^{c+\ell-\sigma}} (c+\ell-\sigma)^{c+\ell-\sigma} e^{-(c+\ell-\sigma)} \int_{1/2}^1 (-\log(1-\xi^2))^{\sigma-1} (1-\xi^2)^c \xi^{-\sigma} d\xi \\ &\leq C \frac{4^{c+\ell-\sigma}}{q_+^{c+\ell-\sigma}} (c+\ell-\sigma)^{c+\ell-\sigma} e^{-(c+\ell-\sigma)} \int_{1/2}^1 (-\log(1-\xi^2))^{\sigma-1} (1-\xi^2)^c \xi d\xi. \end{aligned}$$

Now,

$$\begin{aligned} \int_{1/2}^1 (-\log(1-\xi^2))^{\sigma-1} (1-\xi^2)^c \xi d\xi &\leq C \int_{1/2}^1 (-\log(1-\xi^2))^{\sigma-1/2} (1-\xi^2)^c \xi d\xi \\ &\leq \frac{C}{(c+1)^{\sigma+1/2}} \leq \frac{C}{(c+1)^{1/2}}, \end{aligned}$$

after the change of variable  $1-\xi^2 = e^{-t}$  in the second inequality. Therefore

$$J_2 \leq C \frac{4^{c+\ell-\sigma}}{q_+^{c+\ell-\sigma}} (c+\ell-\sigma)^{c+\ell-\sigma-1/2} e^{-(c+\ell-\sigma)}.$$

Finally, by Stirling's approximation, we conclude the bound for  $J_2$ .  $\square$

*Proof of Lemma 5.3.* The case  $b = 0$  can be seen in [15, Lemma 5.8]. For  $b > 0$ , with the obvious bound

$$\left(\frac{1-s}{A-Bs}\right)^{\alpha+1/2} \leq \frac{1}{A^{\alpha+1/2}}$$

and denoting by  $L$  the integral to be estimated, we have

$$L \leq \frac{1}{A^{\alpha+1/2}} \int_0^1 \frac{(1-s)^{b-1}}{(A-Bs)^{b+\lambda}} ds.$$

Then, the change of variable  $1-s = \frac{A-B}{B}z$  gives

$$L \leq \frac{1}{A^{\alpha+1/2}} \frac{1}{B^b} \frac{1}{(A-B)^\lambda} \int_0^{\frac{B}{A-B}} \frac{z^{b-1}}{(1+z)^{b+\lambda}} dz = \frac{\Gamma(b)\Gamma(\lambda)}{\Gamma(b+\lambda)} \frac{1}{A^{\alpha+1/2}} \frac{1}{B^b} \frac{1}{(A-B)^\lambda}.$$

$\square$

*Proof of Lemma 5.4.* Due to (5.2), it suffices to analyze the integral  $J$  appearing in the proof of Proposition 4.2 with  $\alpha + aj$  instead of  $\alpha_i$  and in the one dimensional case. After the change of variable  $s = 2u - 1$  becomes

$$J = 4^{\alpha+aj} \exp\left(-\frac{(x-y)^2}{4\xi} - \frac{\xi(x+y)^2}{4}\right) \int_0^1 \exp\left(-xyu\left(\frac{1}{\xi} - \xi\right)\right) u^{\alpha+aj-1/2}(1-u)^{\alpha+aj-1/2} du.$$

Now, it is easy to check that

$$\begin{aligned} J &\leq 4^{\alpha+aj+1/2} \exp\left(-\frac{(x-y)^2}{4\xi} - \frac{\xi(x+y)^2}{4}\right) \int_0^{1/2} \exp\left(-xyu\left(\frac{1}{\xi} - \xi\right)\right) u^{\alpha+aj-1/2}(1-u)^{\alpha+aj-1/2} du \\ &\leq C 4^{\alpha+aj+1/2} \exp\left(-\frac{(x-y)^2}{4\xi} - \frac{\xi(x+y)^2}{4}\right) \int_0^{1/2} \exp\left(-xyu\left(\frac{1}{\xi} - \xi\right)\right) u^{\alpha+aj-1/2} du. \end{aligned}$$

When  $aj \geq 1$ , the change of variable  $xyu\left(\frac{1}{\xi} - \xi\right) = z$  gives

$$\begin{aligned} J &\leq 4^{\alpha+aj+1/2} \exp\left(-\frac{(x-y)^2}{4\xi} - \frac{\xi(x+y)^2}{4}\right) \int_0^{1/2} \exp\left(-xyu\left(\frac{1}{\xi} - \xi\right)\right) u^{aj-1} du \\ &\leq 4^{\alpha+aj+1/2} \exp\left(-\frac{(x-y)^2}{4\xi} - \frac{\xi(x+y)^2}{4}\right) (xy)^{-aj} \left(\frac{\xi}{1-\xi^2}\right)^{aj} \Gamma(aj) \\ &\leq C 4^{\alpha+aj+1/2} \exp\left(-\frac{(x-y)^2}{8\xi}\right) |x^2 - y^2|^{-2\alpha-1} (xy)^{-aj} \left(\frac{\xi}{1-\xi^2}\right)^{aj} \Gamma(aj) \end{aligned}$$

where in the last step we have used the estimate

$$\begin{aligned} \exp\left(-\frac{(x-y)^2}{4\xi} - \frac{\xi(x+y)^2}{4}\right) &\leq C \exp\left(-\frac{(x-y)^2}{8\xi}\right) \exp(-c|x^2 - y^2|) \\ &\leq C \exp\left(-\frac{(x-y)^2}{8\xi}\right) |x^2 - y^2|^{-2\alpha-1}. \end{aligned}$$

The result in this case follows from (4.3), (4.6) and (5.2).

The case  $j = 0$  is even easier since

$$\int_0^{1/2} \exp\left(-xyu\left(\frac{1}{\xi} - \xi\right)\right) u^{\alpha-1/2} du \leq C_\alpha,$$

as it is shown in the proof of Proposition 4.2.  $\square$

*Proof of Proposition 5.5.* From the identity  $(L_\alpha)^{-\sigma} = \mathcal{I}_{\alpha,\sigma}$  in  $L^2(\mathbb{R}^+, d\mu_\alpha)$ , the estimate in (5.7) can be deduced from (5.3), (5.4), (5.2), (4.6), and Proposition 4.2.

To obtain the bound in (5.8) we start considering the case  $|x - y| > x/2$ . Proceeding as in the proof of Proposition 4.3, we have that the kernel  $\bar{\mathcal{K}}_{\alpha,\sigma}$  can be estimated by  $e^{c(x-y)^2}(xy)^{-\alpha-1/2}$ , then

$$x^{2\sigma} \int_{|x-y|>x/2} \bar{\mathcal{K}}_{\alpha,\sigma}(x, y) d\mu_\alpha(y) \leq C \int_{|x-y|>x/2} |x - y|^{2\sigma-\alpha-1/2} e^{-c(x-y)^2} y^{\alpha+1/2} dy.$$

If  $0 < y < x/2$ , we have

$$\begin{aligned} \int_{|x-y|>x/2} |x - y|^{2\sigma-\alpha-1/2} e^{-c(x-y)^2} y^{\alpha+1/2} dy &\leq C \int_{|x-y|>x/2} |x - y|^{2\sigma} e^{-c(x-y)^2} dy \\ &\leq C \int_0^\infty z^{2\sigma} e^{-z^2} dz \leq C. \end{aligned}$$

When  $y > 3x/2$ , it is verified that  $|x - y| \sim y$  and so

$$\int_{|x-y|>x/2} |x - y|^{2\sigma-\alpha-1/2} e^{-c(x-y)^2} y^{\alpha+1/2} dy \leq C \int_0^\infty y^{2\sigma} e^{-cy^2} dy \leq C.$$



In the most delicate region  $|x - y| \leq x/2$  we cut the integral in  $\bar{\mathcal{K}}_{\alpha,\sigma}$  in the intervals  $(0, 1/2)$  and  $[1/2, 1)$ . For the second one, the integral of the kernel is controlled by

$$x^{2\sigma} e^{-cx^2} \int_{x/2}^{3x/2} e^{-c(x-y)^2} \int_{1/2}^1 (-\log(1 - \xi^2))^{\sigma-1} (1 - \xi^2)^{-1/2} d\xi dy.$$

This last integral is bounded because the inner integral is smaller than a constant and  $x^{2\sigma} e^{-cx^2} \leq C$ .

In the case  $\xi \in (0, 1/2)$ , after some changes of variables, we have

$$\begin{aligned} x^{2\sigma} \int_{x/2}^{3x/2} \int_0^{1/2} \xi^{\sigma-3/2} \exp\left(-\frac{(x-y)^2}{4\xi} - \frac{\xi x^2}{4}\right) d\xi dy \\ = Cx \int_{x/2}^{3x/2} \int_0^{x^2/2} s^{\sigma-3/2} \exp\left(-\frac{(x(x-y))^2}{4s} - \frac{s}{4}\right) ds dy \\ = C \int_0^{x^2/2} \int_0^{x^2/2} s^{\sigma-3/2} \exp\left(-\frac{z^2}{4s} - \frac{s}{4}\right) ds dz \\ \leq C \int_0^\infty \int_0^\infty s^{\sigma-3/2} \exp\left(-\frac{z^2}{4s} - \frac{s}{4}\right) ds dz \leq C, \end{aligned}$$

where in the last step we have used that

$$\int_0^\infty \int_0^\infty s^{\sigma-3/2} \exp\left(-\frac{z^2}{4s} - \frac{s}{4}\right) ds dz = \int_0^\infty s^{\sigma-1} e^{-s/4} ds \int_0^\infty e^{-t^2/4} dt = 4^\sigma \sqrt{\pi} \Gamma(\sigma).$$

□

## REFERENCES

- [1] A. Bernardis and Ó. Salinas, Two-weight norm inequalities for the fractional maximal operator on spaces of homogeneous type, *Studia Math.* **108** (1994), 201-207.
- [2] B. Bongioanni and J. L. Torrea, Sobolev spaces associated to the harmonic oscillator, *Proc. Indian Acad. Sci.* **116** (2006), 1-24.
- [3] P. Balodis and A. Córdoba, The convergence of multidimensional Fourier-Bessel series, *J. Anal. Math.* **77** (1999), 269-286.
- [4] A. Carbery, E. Romera, and F. Soria, Radial weights and mixed norm inequalities for the disc multiplier, *J. Funct. Anal.* **109** (1992), 52-75.
- [5] Ó. Ciaurri, L. Roncal, and P. R. Stinga, Fractional integrals on compact Riemannian symmetric spaces of rank one, to appear in *Adv. Math.*, [arXiv:1205.3957](#).
- [6] A. Córdoba, The disc multiplier, *Duke Math. J.* **58** (1989), 21-29.
- [7] K. Coulembier, H. De Bie, and F. Sommen, Orthogonality of Hermite polynomials in superspace and Mehler type formulae, *Proc. Lond. Math. Soc.* **103** (2011), 786-825.
- [8] J. Duoandikoetxea, *Fourier analysis*, American Mathematical Society, Providence, 2001.
- [9] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Mathematics Studies **16**, North-Holland Publishing Co., Amsterdam, 1985.
- [10] G. Gasper, K. Stempak, and W. Trebels, Fractional integration for Laguerre expansions, *Methods Appl. Anal.* **2** (1995), 67-75.
- [11] G. Gasper and W. Trebels, Norm inequalities for fractional integrals of Laguerre and Hermite expansions, *Tohoku Math. J.* **52** (2000), 251-260.
- [12] Y. Kanjin and E. Sato, The Hardy-Littlewood theorem on fractional integration for Laguerre series, *Proc. Amer. Math. Soc.* **123** (1995), 2165-2171.
- [13] A. Nowak and L. Roncal, Potential operators associated with Jacobi and Fourier-Bessel expansions, preprint, 2012.
- [14] A. Nowak and K. Stempak, Negative powers of Laguerre operators, *Canad. J. Math.* **64** (2012), 183-216.
- [15] A. Nowak and K. Stempak, Nowak, Riesz transforms for multi-dimensional Laguerre function expansions, *Adv. Math.* **215** (2007), 642-678.
- [16] N. N. Lebedev, *Special functions and its applications*, Dover, New York, 1972.
- [17] B. Muckenhoupt and E. M. Stein, Classical expansions and their relation to conjugate harmonic functions, *Trans. Amer. Math. Soc.* **118** (1965), 17-92.
- [18] J. L. Rubio de Francia, Transference principles for radial multipliers, *Duke Math. J.* **58** (1989), 1-19.
- [19] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, Princeton University Press, Princeton, 1970.
- [20] E. M. Stein and G. Weiss, Fractional integrals on  $n$ -dimensional Euclidean space, *J. Math. Mech.* **7** (1958), 503-514.

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